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UNSTEADY GAS DIFFUSION ACROSS A TWO-DIMENSIONAL PERMEABLE CAVITY SURFACE

K. Ravindra

Technical Memorandum
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The Pennsylvania State University
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From: K. Ravindra

Subject: <u>Unsteady Gas Diffusion across a Two-Dimensional Permeable Cavity Surface</u>

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LIST OF SYMBOLS

A_T - exchange coefficient

$\dot{b}$ - characteristic length

c - concentration difference, $c = c(x,y,t)$

$\bar{c}$ - time mean concentration difference, $\bar{c} = \bar{c}(x,y,t)$

C - absolute concentration difference at a field point

C_∞ - concentration of dissolved gases in the free stream

C_o - concentration on the permeable surface

c_o - $C_o - C_\infty$

f - source strength distribution function, frequency of oscillation, Hertz

F - Laplace transform of function f

I_o - modified Bessel's function of zeroth order, second kind

j - $\sqrt{-1}$

K_o - modified Bessel's function of zeroth order, first kind

k - reduced frequency

ℓ - non-dimensional cavity length, $\ell = \ell(t)$

ℓ_o - steady cavity length

P_G - cavity gas pressure

P_{FS} - partial pressure of gas in the free stream

r - non-dimensional radial coordinate

R - non-dimensional number

s - Laplace variable

t - time

u - instantaneous velocity fluctuation in the X-direction

U_∞ - velocity in the X direction

LIST OF SYMBOLS [continuation]

v - instantaneous velocity fluctuation in the Y direction

w - instantaneous velocity fluctuation in the Z direction

x - non-dimensional X coordinate

y - non-dimensional Y coordinate

z - non-dimensional Z coordinate

α_1 - dissolved gas content in ppm

β - Henry's law constant

ϵ - amplitude of cavity oscillation, $\epsilon = \epsilon(\omega,\sigma)$

ϕ - phase angle between body motion and cavity motion

κ - molecular mass diffusivity

ν - turbulent eddy mass diffusivity

π - 3.1415 . . .

ρ - density

σ - cavitation number

INTRODUCTION

Recent experiments on cavity pressure measurement [2,3] have demonstrated that the cavity pressure inside a cavity varies with cavity oscillations and that the cavity pressure fluctuations are significant at low frequencies. The gaseous diffusion across a cavity surface naturally influences the cavity pressure and hence the characteristics of the cavity (viz., size, shape and behavior) to a great extent. While previous investigations have addressed the steady gaseous diffusion [1,4] across cavity surfaces, a corresponding analysis for unsteady diffusion in an unsteady cavity flow is, to the best of our knowledge, non-existent.

Unsteady gaseous diffusion into a cavity may result from two distinct mechanisms. First, since the cavity length fluctuates, the gaseous diffusion into the cavity fluctuates correspondingly. Second, since the cavity gas pressure itself fluctuates, the concentration gradient across the cavity surface is a function of time. In this manuscript, we formulate and solve analytically, the unsteady gaseous diffusion across a two-dimensional unsteady cavity surface. The results of this analysis will give a better understanding of the parameters that affect the unsteady gaseous diffusion across a cavity surface.

Assumptions

It is assumed in the foregoing analysis that the flow is incompressible, two-dimensional, isotropic and turbulent. [Turbulence is inherently three-dimensional in nature. Nonetheless, a two-dimensional flow field with unit depth may be considered for diffusion analysis.] Boussinesq's hypothesis [4] is used in turbulence modeling and G. I. Taylor's [5] statistical theory of turbulence is used in estimating the mass diffusivity for the flow field. It is assumed that the mechanism for turbulent momentum transfer and turbulent concentration transfer are identical. It is also assumed that the gaseous diffusion occurs instantaneously across the cavity surface and that the concentration gradient on the cavity surface fluctuates in phase with the pressure fluctuations inside the cavity.

Theoretical Analysis

Gaseous diffusion will occur when there exists a dissolved-gas concentration difference between the free stream and the liquid on the cavity surface. If α_1 denotes the dissolved gas content in the free stream in parts per million, by moles, then by Henry's law, the maximum partial pressure of gas in the free stream P_{FS} is

$$P_{FS} = \alpha_1 \beta \quad , \tag{1}$$

where β is the Henry's law constant. If $P_G(t)$ is the instantaneous partial pressure of non-condensable gas in the cavity, then the mean concentration difference c(t) expressed in moles is

$$c(t) = (\alpha_1 - \frac{P_G}{\beta}) \quad . \tag{2}$$

We assume, as is customary, that the mechanism for turbulent diffusion of gas in the liquid and across the wall and the turbulent momentum transfer are similar and that gradients in mean velocity can be neglected. Then, the turbulent diffusion is due solely to the gradient of the mean concentration c(x,y). Consider a turbulent flow field having a uniform mean velocity U_∞ in the positive X direction, as shown in Figure 1. Let the instantaneous concentration at a point (x,y) be C(x,y,t). Also, let C_∞ be the concentration at large distances from the origin and C_0 be the saturation concentration on the cavity wall. Let $c = C - C_\infty$ be the concentration difference. From Fick's law of diffusion and conservation of mass (see, for example, Batchelor [6]) it follows that

$$(U \cdot \nabla)c + \frac{\partial c}{\partial t} = \kappa \nabla^2 c \tag{3}$$

where ∇^2 denotes the Cartesian Laplacian operator and κ is the molecular diffusivity.

Let the instantaneous velocity components be represented by $(U_\infty + u')$, v' and w' in the three orthogonal directions and let $c = \bar{c} + c'$ where $\bar{c}$ is the time mean concentration difference. The components u', v' and w' are the instantaneous velocity fluctuations and c' is the instantaneous concentration difference. Then, Eq. (3) becomes (see, for example, Goldstein [7])

$$\frac{\partial \bar{c}}{\partial t} + U_\infty \frac{\partial \bar{c}}{\partial x} - \kappa \nabla^2 \bar{c} + \frac{\partial c'}{\partial t} + [U_\infty \frac{\partial c'}{\partial x} + u' \frac{\partial \bar{c}}{\partial x} + u' \frac{\partial c'}{\partial x} + v' \frac{\partial \bar{c}}{\partial y} + v' \frac{\partial c'}{\partial y}$$

$$+ w' \frac{\partial \bar{c}}{\partial z} + w' \frac{\partial c'}{\partial z}] - \kappa \nabla^2 c' = 0 \quad . \tag{4}$$

Time-averaging the above equation, we obtain

$$\frac{\partial \bar{c}}{\partial t} + U_\infty \frac{\partial \bar{c}}{\partial x} = \frac{\partial}{\partial x} (\kappa \frac{\partial \bar{c}}{\partial x} - \overline{u'c'}) + \frac{\partial}{\partial y} (\kappa \frac{\partial \bar{c}}{\partial y} - \overline{u'c'}) + \frac{\partial}{\partial z} (\kappa \frac{\partial \bar{c}}{\partial z} - \overline{w'c'}) \tag{5}$$

where the overbar indicates time averaging. Equation (5) shows that in turbulent flows the eddy concentration transport terms $-\overline{u'c'}$, $-\overline{v'c'}$ and $-\overline{w'c'}$ add to the molecular concentration transports $\kappa \frac{\partial \bar{c}'}{\partial x}$, $\kappa \frac{\partial \bar{c}}{\partial y}$ and $\kappa \frac{\partial \bar{c}}{\partial z}$ respectively. The eddy transport terms are large compared to molecular transport terms (7). Therefore, the latter are neglected and Equation (5) reduces to

$$\frac{\partial \bar{c}}{\partial t} + U_\infty \frac{\partial \bar{c}}{\partial x} = - \frac{\partial}{\partial x} (\overline{u'c'}) - \frac{\partial}{\partial y} (- \overline{u'c'}) - \frac{\partial}{\partial z} (- \overline{w'c'}) \quad . \tag{6}$$

We now restrict our study to two-dimensional flows. We represent the turbulent shear stress $\rho\overline{u'c'}$ and $\rho\overline{u'c'}$ using Boussinesq's hypothesis as

$$- \rho\overline{u'c'} = A_T \frac{\partial \bar{c}}{\partial x} , \tag{7a}$$

and

$$- \rho\overline{v'c'} = A_T \frac{\partial \bar{c}}{\partial y} , \tag{7b}$$

where A_T is the exchange coefficient or eddy mass conductivity. Let ν denote the eddy mass diffusivity. Then,

$$\nu = A_T/\rho \quad , \tag{8}$$

where ρ is the density of the liquid. Substitution of Equations (7) and (8) in (6) yields

$$\frac{1}{\nu} \frac{\partial \bar{c}}{\partial c} + \frac{U_\infty}{\nu} \frac{\partial \bar{c}}{\partial x} = \frac{\partial^2 \bar{c}}{\partial x^2} + \frac{\partial \bar{c}}{\partial y^2} \tag{9}$$

We normalize the distances along x and y directions by a characteristic length $\bar{b}$ and the time by a reference time $\bar{b}/U_\infty$, such that

$$\left.\begin{aligned} x' &= \frac{x}{\bar{b}} && (10a)\\ y' &= \frac{y}{\bar{b}} && (10b)\\ t' &= \frac{t}{(\bar{b}/U_\infty)} && (10c)\end{aligned}\right\} .$$

Then, Equation (9) may be written as

$$\frac{\partial \bar{c}}{\partial t} + \frac{\partial \bar{c}}{\partial x} = \frac{1}{2R}\left[\frac{\partial^2 \bar{c}}{\partial x^2} + \frac{\partial^2 \bar{c}}{\partial y^2}\right] , \qquad (11)$$

where the quantity R is a nondimensional number given by

$$R = \frac{U_\infty \bar{b}}{2\nu} , \qquad (12)$$

and the quantities x, y and t are now nondimensional.

The boundary conditions pertinent to the two-dimensional unsteady diffusion are:

$$\frac{\partial \bar{c}}{\partial y} = 0 \quad ; \quad y = 0 \ , \ x \le 0 \ , \ x \ge \ell(t) \qquad (13a)$$

and

$$\bar{c} = c_o(t) \quad ; \quad y = 0 \ , \ 0 < x < \ell(t) \qquad (13b)$$

where $\ell(t)$ is the unsteady cavity length normalized with respect to the characteristic length $\bar{b}$ and is given by

$$\ell(t) = \ell_o + \epsilon \sin(kt + \phi) . \qquad (14)$$

In Equation (14), ℓ_o is the steady cavity length, ϵ is the amplitude of cavitation oscillation, $\epsilon = \epsilon(k,\sigma)$ and k is the reduced frequency defined by

$$k = (2\pi f)\bar{b}/U_\infty \quad , \tag{15}$$

where f is the frequency of body oscillation in Hertz. ϕ is the phase angle between the body motion and the cavity motion, $\phi = \phi(k)$. The solution of Equation (11) subject to the prescribed boundary conditions (13) is obtained in stages. First, a point source solution that has the proper radial symmetry and behavior is obtained. Second, the point source solution is used to formulate an integral equation relating the concentration at a field point and the mass flux per unit length along the cavity. Third, the concentration difference at the cavity surface is used to solve the integral equation.

The fundamental solution to Equation (11) for an oscillating unit source at the origin is [see Appendix C-1]

$$\bar{c}(x,y,t) = \frac{1}{2\pi A_T} K_o(R\sqrt{x^2 + y^2})e^{Rx}e^{j(kt+\phi-\omega x)} \quad . \tag{16}$$

Let f(x,t) be the instantaneous mass flux per unit length between x = 0 and x = $\ell(t)$ The the function $\bar{c}(x,y,t)$ can be expressed in terms of f(x,t) as follows:

$$\bar{c}(x,y,t) = \frac{e^{j\phi}}{2\pi A_T} \int_0^{\ell(t)} f(\xi,t)e^{R(x-\xi)} K_o(R\sqrt{(x-\xi)^2 + y^2})e^{jk(t-x+\xi)}d\xi \tag{17}$$

If we now require that $\bar{c}(x,0,t) = c_o(t)$ in the interval $0 < x < \ell(t)$, the source strength is determined from the integral equation

$$c_o = \frac{e^{j(kt+\phi)}}{2\pi A_T}\left[\int_0^x f(\xi,t)e^{R(x-\xi)}K_o(R(x-\xi))e^{-jk(x-\xi)}d\xi\right.$$

$$\left.+\int_x^{\ell(t)} f(\xi,t)e^{-R(\xi-x)}K_o(R(\xi-x))e^{jk(\xi-x)}d\xi\right] \quad , \qquad (18)$$

where the positive branch of the square root has been taken. The function $c(x,y,t)$ also satisfies the conditions that $c \to 0$ as $(x^2 + y^2) \to \infty$ and that $\frac{\partial c(x,0)}{\partial y} = 0$ when $x < 0$ and $x > \ell(t)$.

In order to determine $f(x,t)$, we shall make use of the fact that R is a large number for the present study. Therefore, K_o in Equation (18) can be replaced by the first term of its asymptotic expansion,

$$K_o(z) \approx e^{-z}\sqrt{\frac{\pi}{2z}} \qquad (19)$$

A further simplification can be obtained in the second integral in Equation (18) by noting that the strong negative exponential will cause $f(\xi,t)$ to be a slowly varying function compared to the term $e^{-R(\xi-x)}K_o(R(\xi - x))$, so that $f(\xi,t)$ contributes to the integration only near $\xi = x$. Therefore, we replace the second integral in Equation (18) by the approximate value,

$$\frac{\pi}{2R} f(x,t) \frac{\sqrt{\pi}}{\sqrt{2R - jk}}$$

With these approximations, Equation (18) may be written as

$$c_o = \frac{\pi}{2R} \frac{e^{j(kt+\phi)}}{2\pi A_T} \left[\int_0^x \frac{f(\xi,t) e^{-jk(x-\xi)}}{\sqrt{x - \xi}} d\xi + \frac{f(x,t)\sqrt{\pi}}{\sqrt{2R - jk}} \right] . \tag{20}$$

If the Laplace Transform of f(x,t) is denoted by F(s,t), we can transform Equation (20) to find

$$F(s,t) = \frac{2A_T\sqrt{2R}\, c_o e^{-j(kt+\phi)}}{s\left[\frac{1}{(s + jk)^{1/2}} + \frac{1}{(2R - jk)^{1/2}}\right]} . \tag{21}$$

On the cavity surface, from Equation (C.1.2),

$$c_o(t) = c_1(x,0) e^{j(kt+\phi)} .$$

Equation (21) may be rearranged to read

$$F(s) = 2A_T \sqrt{2R}\, c_1 \frac{\sqrt{2R - jk}\sqrt{s + jk}}{s[\sqrt{2R - jk} + \sqrt{s + jk}]} . \tag{22}$$

The inverse of this transform gives [see Appendic C.2] for f(x) the result

$$f(x) = Q \left[\frac{a}{\alpha} - \frac{b^2}{\alpha} e^{-\alpha x} \mathrm{erfc}\, b\sqrt{x} - \frac{b\sqrt{a}}{\alpha} \mathrm{erf}\sqrt{ax}\right] \tag{23}$$

where

$$Q = 2A_T \sqrt{2R}\, c_1 \sqrt{2R - jk} \tag{24b}$$

$$a = jk \tag{24b}$$

$$b = \sqrt{2R - jk} \tag{24c}$$

$$\alpha = (2jk - 2R) \tag{24d}$$

and erf(x) and erfc(x) denote the error function and complimentary error function respectively. We note here the fact that the function f(x) which represents the mass flux per unit length is independent of time.

The rate at which the mass is diffused per unit widt. along the entire length of the permeable interval is obtained from

$$\frac{dM}{dt} = \int_0^{\ell(t)} f(x,t)dx \tag{25}$$

i.e.,

$$\frac{dM}{dt} = Q \int_0^{\ell(t)} \left(\frac{a}{\alpha} - \frac{b^2}{\alpha} e^{-\alpha x} \mathrm{erfc}\, b\sqrt{x} - \frac{b\sqrt{a}}{\alpha} \mathrm{erf}\sqrt{ax}\right)dx \tag{26}$$

The integral in Equation (26) has been evaluated in Appendix C.3 and the result is

$$\frac{dM}{dt} = \frac{Q}{\alpha} \left[a\ell + \frac{b^2}{\alpha} e^{-\alpha\ell} \mathrm{erfc}\, b\sqrt{\ell} + \frac{b^3}{\alpha\sqrt{a}} \mathrm{erf}\sqrt{a\ell} - \frac{b^2}{\alpha} - b\sqrt{a\ell}\, \mathrm{erf}\sqrt{a\ell}\right.$$

$$\left. - b\sqrt{\frac{\ell}{\pi}} e^{-a\ell} + \frac{b}{2\sqrt{a}} \mathrm{erf}\sqrt{a\ell}\right] . \tag{27}$$

Equation (27) gives the instantaneous mass flow rate across a two-dimensional cavity. In order to utilize this equation, the variation of cavity length as a function of frequency of oscillation must be established [see, for example, Chapter 3 of Ref. (3)]. In the limit, when the reduced frequency k is zero, it can be shown that the expression for mass flow rate reduces to the steady state mass flow rate established by Parkin (1).

Conclusions

We have developed an analytical model for the prediction of gaseous diffusion across a two-dimensional unsteady cavity surface. This model takes into account the change in cavity length as well as changes in cavity pressure in predicting the gas diffusion across the cavity surface. The expressions for mass flux per unit length and mass diffusion rate across the entire cavity length reduce to those obtained by Parkin (8) when the reduced frequency k is zero. We reiterate here the fact that this unsteady gaseous diffusion analysis is valid only for harmonic variations in cavity length and cavity gas pressure. Nonetheless, this analysis can easily be extended to encompass a general nonharmonic motion of the cavity by Fourier representation of the cavity length and cavity gas pressure fluctuations.

APPENDIX C.1

Fundamental solution to Equation (11) for an oscillating unit source at the origin:

Equation (4.11) is

$$\frac{\partial \bar{c}}{\partial t} + \frac{\partial \bar{c}}{\partial x} = \frac{1}{2R}\left[\frac{\partial^2 \bar{c}}{\partial x^2} + \frac{\partial^2 \bar{c}}{\partial y^2}\right] \quad . \tag{C.1.1}$$

Since (t) is periodic, we assume $\bar{c}(x,y,t)$ to be periodic and write

$$\bar{c}(x,y,t) = c_1(x,y)e^{j(kt+\phi)} \quad , \tag{C.1.2}$$

where ϕ is the phase angle between the motion of the body and the motion of the cavity and $j = \sqrt{-1}$. It is to be noted that only the real part of the analysis that follows is pertinent. Substituting Equation (C.1.2) in (C.1.1), we obtain

$$2c_1 jkR + 2R\frac{\partial c_1}{\partial x} = \nabla^2 c_1 \quad . \tag{C.1.3}$$

To solve Equation (C.1.3) we replace the Laplacian on the right-hand side by its counterpart in plane polar coordinates when variations only in the radial direction are permitted. Then,

$$2c_1 jkR + 2R\frac{\partial c_1}{\partial x} = \frac{\partial^2 c_1}{\partial r^2} + \frac{1}{r}\frac{\partial c}{\partial r} \quad , \tag{C.1.4}$$

where

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ \theta &= \tan^{-1}\frac{y}{x} \end{aligned}\right\} . \tag{C.1.5}$$

By letting

$$\frac{x}{2R} = s \quad , \tag{C.1.6}$$

Equation (C.1.4) can be written as

$$2c_1 jkR + \frac{\partial c_1}{\partial s} = \frac{\partial^2 c_1}{\partial r^2} + \frac{1}{r}\frac{\partial c}{\partial r} \quad . \tag{C.1.7}$$

We now use separation of variables to solve Equation (C.1.7). By substituting

$$c_1(s,r) = S(s)X(r) \quad , \tag{C.1.8}$$

Equation (C.1.7) may be written as

$$2jkR + \frac{S'}{S} = \frac{X''}{X} + \frac{1}{r}\frac{X'}{X} = \kappa_1^2 \quad , \tag{C.1.9}$$

where κ_1^2 is an unknown constant. Therefore,

$$X'' + \frac{X'}{r} - \kappa_1^2 X = 0 \quad , \tag{C.1.10}$$

and

$$\frac{S'}{S} = \kappa_1^2 - 2jkR \quad . \tag{C.1.11}$$

Equation (C.1.10) may be re-written as

$$r^2 \frac{d^2X}{dr^2} + r\frac{dX}{dr} - \kappa_1^2 r^2 X = 0 \quad , \tag{C.1.12}$$

and the solution for $\kappa_1^2 > 0$ is

$$X(r) = A_1 I_0(\kappa_1 r) + A_2 K_o(\kappa_1 r) \quad , \tag{C.1.13}$$

where A_1 and A_2 are the constants and K_o and I_0 are the modified Bessel's function of zeroth order of first and second kind respectively.

Note 1: Negative values of κ_1^2 yield functions J_0 and Y_0 (Bessel's function of zeroth order of first and second kind respectively) which do not satisfy the boundary conditions.

Note 2: $\kappa_1^2 = 0$ yields a logarithmic function which again does not satisfy the boundary conditions.

For large distances from the origin, the function $c_1(s,r)$ should vanish. Hence, we set $A_1 = 0$ in Equation (C.1.13). Therefore,

$$X(r) = A_2 K_o(\kappa_1 r) \quad . \tag{C.1.14}$$

The solution of Equation (C.1.11) is easily written as

$$S(s) = Be^{(\kappa_1^2 - 2jkR)s} \quad , \tag{C.1.15}$$

where B is an unknown constant. Hence from Equation (C.1.8),

$$C_1(s,r) = B_1 e^{(\kappa_1^2 - 2jkR)s} K_o(\kappa_1 r) \qquad (C.1.16)$$

where $B_1 = BA_2$ is a new unknown constant. From Equation (C.1.2),

$$\bar{c}(s,r,t) = B_1 K_o(\kappa_1 r) e^{\kappa_1^2 s} e^{j(k+\phi-2kRs)} \quad . \qquad (C.1.17)$$

The constants B_1 and κ_1 evaluated using the boundary conditions are found to be

$$B_1 = \frac{1}{2\pi A_T} \quad \text{and} \quad \kappa_1 = R \quad .$$

Hence

$$\bar{c}(x,y,t) = \frac{1}{2\pi A_T} K_o(R\sqrt{x^2+y^2}) e^{Rx} e^{j(kt+\phi-\omega x)} \quad . \qquad (C.1.18)$$

APPENDIX C.2

To obtain the inverse Laplace Transform of the function

$$F(s,t) = \frac{Q\sqrt{s + jk}}{s[\sqrt{2R - jk} + \sqrt{s + jk}} \quad . \tag{C.2.1}$$

Let

$$a = jk \tag{C.2.2}$$

$$b = \sqrt{2R - jk} \tag{C.2.3}$$

$$F(s,t) = \frac{Q\sqrt{s + a}}{s[\sqrt{s + a} + b]} \tag{C.2.4}$$

$$F(s,t) = Q[\frac{\sqrt{s + a} + b}{s[\sqrt{s + a} + b]} - \frac{b}{s[\sqrt{s + a} + b]}] \tag{C.2.5}$$

$$F(s,t') = Q[\frac{1}{s} - \frac{b}{s[\sqrt{s + a} + b]}] \quad . \tag{C.2.6}$$

Let

$$H(s) = \frac{b}{s[\sqrt{s + a} + b]} \tag{C.2.7}$$

or

$$H(s) = \frac{b[\sqrt{s + a} - b]}{s[s + a - b^2]} \quad .$$

Denote

$$\alpha = a - b^2 \quad . \tag{C.2.8}$$

Then,

$$H(s) = \frac{b\sqrt{s + a}}{s(s + \alpha)} - \frac{b^2}{s(s + \alpha)} \tag{C.2.9}$$

Denote

$$Z(s) = \frac{b\sqrt{s + a}}{s(s + \alpha)} \tag{C.2.10}$$

$$\frac{Z(s)}{b} = \frac{s + \alpha + b^2}{s(s + \alpha)\sqrt{s + a}}$$

$$\frac{Z(s)}{b} = \frac{1}{s\sqrt{s + a}} + \frac{b^2}{s(s + \alpha)\sqrt{s + a}}$$

$$\frac{Z(s)}{b} = \frac{1}{s\sqrt{s + a}} + \frac{b^2}{\alpha}\left(\frac{1}{s\sqrt{s + a}} - \frac{1}{(s + a)\sqrt{s + a}}\right) \tag{C.2.11}$$

Substituting Equation (C.2.11) in (C.2.9), one has

$$H(s) = \frac{b}{s\sqrt{s + a}} + \frac{b^3}{\alpha s\sqrt{s + a}} - \frac{b^3}{\alpha}\,\frac{1}{(s + \alpha)\sqrt{s + a}} - \frac{b^2}{s(s + \alpha)} \quad . \tag{C.2.12}$$

Substituting Equation (C.2.12) in Equation (C.2.6), one gets

$$F(s) = Q[\frac{1}{s} - \frac{b}{s\sqrt{s + a}} - \frac{b^3}{\alpha s\sqrt{s + a}} + \frac{b^3}{\alpha(s + \alpha)\sqrt{s + a}} + \frac{b^2}{s(s + \alpha)}] \quad . \tag{C.2.13}$$

Taking the inverse Laplace Transform on both sides, one finds

$$f(x) = Q[\frac{a}{\alpha} - \frac{b^2}{\alpha}e^{-\alpha x}\mathrm{erfc}\, b\sqrt{x} - \frac{b\sqrt{a}}{\alpha}\mathrm{erf}\sqrt{ax}] \quad . \tag{C.2.14}$$

APPENDIX C.3

Integral Evaluation

Equation (4.25) is

$$\frac{dM}{dt} = \int_0^{\ell(t)} f(x,t)dt \quad , \tag{C.3.1}$$

where

$$f(x,t) = \frac{Q}{\alpha} (a - b^2 e^{-\alpha x} \operatorname{erfc} b\sqrt{x} - b\sqrt{a} \operatorname{erf}\sqrt{ax}) \quad . \tag{C.3.2}$$

Substituting Equation (C.3.2) in (C.3.1),

$$\frac{dM}{dt} = \frac{Q}{\alpha} \left[\int_0^{\ell(t)} a dx - b^2 \int_0^{\ell(t)} e^{-\alpha x} \operatorname{erfc} b\sqrt{x}\, dx - b\sqrt{a} \int_0^{\ell(t)} \operatorname{erf}\sqrt{ax}\, dx \right] \quad . \tag{C.3.3}$$

Let

$$I_1 = \int_0^{\ell(t)} e^{-\alpha x} \operatorname{erfc} b\sqrt{x}\, dx \quad , \tag{C.3.4}$$

and

$$I_2 = \int_0^{\ell(t)} \operatorname{erf}\sqrt{ax}\, dx \quad . \tag{C.3.5}$$

The integral in Equation (C.3.4) may be evaluated by parts.

$$I_1 = \left[\operatorname{erfc} b\sqrt{x}\, \frac{e^{-\alpha x}}{-\alpha}\right]_0^{\ell(t)} + \frac{1}{\alpha} \int_0^{\ell(t)} e^{-\alpha x}(-1)\, \frac{b}{\sqrt{\pi}}\, \frac{e^{-b^2 x}}{\sqrt{x}}\, dx$$

$$I_1 = \frac{1}{\alpha} - \frac{e^{-\alpha \ell} \operatorname{erfc} b\sqrt{\ell}}{\alpha} - \frac{b}{\alpha\sqrt{\pi}} \int_0^{\ell(t)} \frac{e^{-ax}}{\sqrt{x}}\, dx \quad . \qquad (C.3.6)$$

By setting $ax = \eta^2$ in Equation (C.3.6), we can show that

$$I_1 = \frac{1}{\alpha} - \frac{e^{-\alpha \ell} \operatorname{erfc} b\sqrt{\ell}}{\alpha} - \frac{b}{\alpha\sqrt{a}} \operatorname{erf}\sqrt{a\ell} \quad . \qquad (C.3.7)$$

The integral in Equation (C.3.5) can be obtained by parts,

$$I_2 = \left[x \operatorname{erf}\sqrt{ax}\right]_0^{\ell(t)} - \int_0^{\ell(t)} x\, \frac{\sqrt{a}}{\sqrt{\pi}}\, \frac{e^{-ax}}{\sqrt{x}}\, dx$$

i.e.,

$$I_2 = \ell \operatorname{erf}\sqrt{a\ell} - \frac{\sqrt{a}}{\sqrt{\pi}} \int_0^{\ell(t)} \sqrt{x}\, e^{-ax} dx \quad . \qquad (C.3.8)$$

The integral in Equation (C.3.8) may be evaluated again by parts and the result is

$$I_2 = \ell \operatorname{erf}\sqrt{a\ell} + \sqrt{\frac{\ell}{\pi a}}\; e^{-a\ell} - \frac{1}{2a} \operatorname{erf}\sqrt{a\ell} \quad . \qquad (C.3.9)$$

Hence from Equation (C.3.3),

$$\frac{dM}{dt} = \frac{\Omega}{\alpha} [a\ell - \frac{b^2}{\alpha} + \frac{b^2}{\alpha} e^{-\alpha\ell} \mathrm{erfc}\, b\sqrt{\ell} + \frac{b^3}{\alpha\sqrt{a}} \mathrm{erf}\sqrt{a\ell}$$

$$+ \frac{b}{2\sqrt{a}} \mathrm{erf}\sqrt{a\ell} - b\sqrt{\frac{\ell}{\pi}}\, e^{-a\ell} - b\sqrt{a}\, \ell\, \mathrm{erf}\sqrt{a\ell}] \quad . \tag{C.3.10}$$

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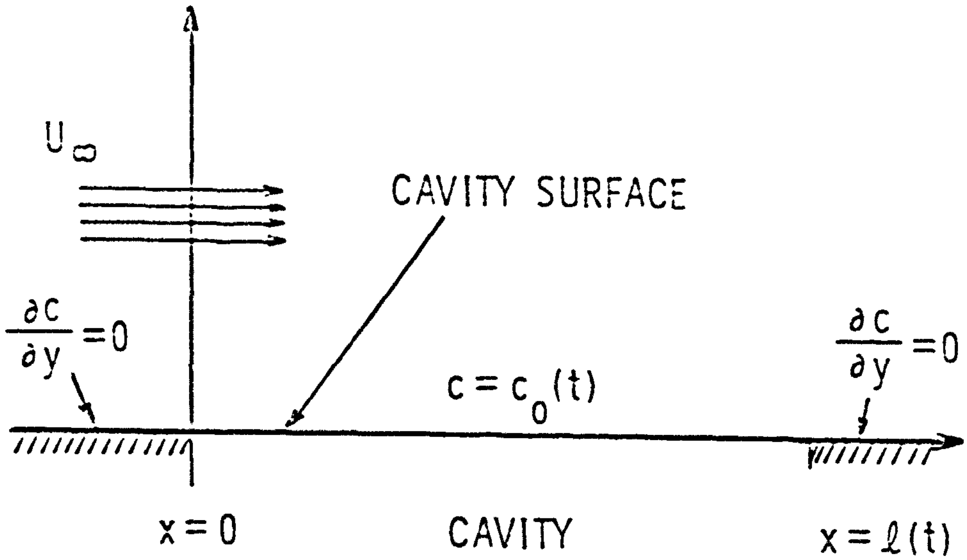

Figure 1. Boundary Conditions for Unsteady Diffusion Analysis.

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